One-dimensional diffusion processes and their boundaries

Inge Helland*

December 2, 1996

Abstract

It is recalled how one-dimensional homogeneous diffusion processes can be constructed from the Wiener process via a time change and a space transformation. No Lipschitz requirements of the drift coefficient and of the diffusion coefficient as functions of the space variable are needed for this construction to be valid. The process constructed in this way will be the unique weak solution of the corresponding stochastic differential equation. Furthermore, a complete classification of boundary types and boundary behaviour is a direct result of the construction. The boundary behaviour of one-dimensional diffusion processes is illustrated by examples, in particular this boundary behaviour is discussed for a population model recently proposed by Lungu and Øksendal.

1 Introduction

We will here be concerned with one-dimensional homogeneous diffusion processes, essentially processes that are solutions of stochastic differential equations with time-homogeneous coefficients

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t. \tag{1}$$

Here W_t is a Wienerprocess, and the statespace of X_t is assumed to be a finite or infinite interval.

*Department of Mathematics, University of Oslo, Box 1053 Blindern, N-0316 Oslo, Norway. (ingeh@math.uio.no)

These processes will be continuous Markov processes, and under weak regularity conditions the *drift coefficient* $\mu(x)$ will have the interpretation

$$\mu(x) = \lim_{h\downarrow 0} h^{-1}\mathrm{E}^x\{X_h - x\}, \tag{2}$$

and the *diffusion coeffisient* $\sigma(x)$ satisfies

$$\sigma^2(x) = \lim_{h\downarrow 0} h^{-1}\mathrm{E}^x\{(X_h - x)^2\} = \lim_{h\downarrow 0} h^{-1}\mathrm{Var}^x\{X_h\}. \tag{3}$$

For later purposes it is convenient to weaken the conditions (2) and (3) slightly and impose insted the three conditions

$$h^{-1}\mathrm{E}^x((X_h - x)I(|X_h - x| \leq 1)) \to \mu(x), \tag{4}$$

$$h^{-1}\mathrm{E}^x((X_h - x)^2 I(|X_h - x| \leq 1)) \to \sigma^2(x) \tag{5}$$

and

$$h^{-1}\mathrm{P}^x(|X_h - x| > \epsilon) \to 0 \text{ for all } \epsilon > 0. \tag{6}$$

The extra condition (6) may be coupled to the continuity of sample paths, and it implies that the truncation constant 1 in (4) and (5) may be replaced by any other constant. If (6) is strengthened to a Lindeberg type condition is:

$$h^{-1}\mathrm{E}^x((X_h - x)^2 I(|X_h - x| > \epsilon)) \to 0, \tag{7}$$

then the indicators may be dropped from (4) and (5).

From an applied point of view the study of this class of processes can be motivated in several ways; first of all by the fact that several real processes can be modelled to some degree of approximation in this way. Another motivation for equations of the form (1) is that it is conceptually natural to look upon many data-generating mechanisms as having arisen through a signal, connected to an expectation, plus noise, connected to a Gaussian variable with some variance parameter. In fact, this motivation is similar to the one that is sometimes given to a completely different class of statistical models, namely the linear normal models. The mathematics behind these two classes of models is very different, of course, but even here there are similarities, for instance connected to the use of linear projections in the inference phase.

During the recent decades important progress in the understanding of diffusion process have been reached using mathematical tools from operator theory, from martingale theory and not least from the general theory of stochastic differential equations. The strength of the latter is that it easily generalizes to multidimensional processes and to processes inhomogeneous in time. One weakness is that μ and σ are required to be Lipschitz continuous ($|\mu(x) - \mu(z)| \leq K|x - z|$ and $|\sigma(x) - \sigma(z)| \leq$

$K|x - z|$ for some K). For a broad, but at the same time rigorous, introduction to the theory of stochastic differential equations, see Øksendal (1995).

It does not seem to be too well appreciated today among all workers in the field that there in the homogeneous one-dimensional case also exists a complete theory of processes essentially of the form (1) where one does not need to assume any Lipschitz type condition for the coeffisients (in fact more general processes than those described by (1) are covered by the theory), and where one can give a complete description of every possible behavior of the process at the boundary points of the state interval. This theory dates back to Feller (1952, 1954, 1957) and it can be found in various disguises in Itô-McKean (1965), Dynkin (1965) and in the last chapter of Breiman (1968). The purpose of the present paper is to give an easily accessible survey of the results of this theory with some illustrations, motivated by the feeling that these results certainly must have potential applications. The main mathematical tools used will be simple probabilistic arguments together with derivation of and solution of ordinary linear differential equations. While there will be some lack of technical rigour at some places, this can always be repaired by refering to one of the books cited. We will try to give the essential part of the arguments for the boundary classification results, but will not give complete proofs. As motivations behind these arguments we will start at the Kolmogorov equations and related equations, but the main building block behind the argument themselves will be the important result that all (time-homogeneous) one-dimensional diffusions can be constructed from the Wiener process via a time change and a space transformation. This latter result is in fact valid for essentially all continuous strong Markov processes on the line with time-homogeneous transitions, a class of processes which includes diffusions with simple discontinuities in $\mu(x)$ or in $\sigma(x)$, for instance.

The motivation behind this paper was a question raised by Bernt Øksendal in the investigation Lungu and Øksendal (1996). This concrete example is treated in some detail in Section 7.

2 Kolmogorov type equations

For simplicity we will assume in this paper that μ and σ are continuous. No Lipschitz condition is assumed, but we assume in this Section that there exists a process (X_t) satisfying the conditions (4), (5) and (6). An explicit construction of such a process will be given in the next Section. Take $X_0 = x$ for this process.

Kolmogorov's backward equation for the density $p(t, x, y)$ of X_t, given $X_0 = x$, is well known:

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2(x)\frac{\partial^2 p}{\partial x^2} + \mu(x)\frac{\partial p}{\partial x}. \tag{8}$$

The initial condition is $p(0,x,y) = \delta(y-x)$, where $\delta(\cdot)$ is Dirac's delta function. The same equation with initial conditions $v(0,x) = g(x)$ is satisfied by $v(t,x) = \mathrm{E}^x(g(X_t))$. One way of solving the equation for p is by introducing the Laplace transform $R(\lambda, x, y) = \int_0^\infty e^{-\lambda t} p(t,x,y)dt$.

For our purpose it is more important that one can find explicit expressions for exit probabilities and for expected exit times from intervals. Let $[a,b]$ be a fixed interval and start the process in $X_0 = x \in (a,b)$. We will find the probability $p_+(x)$ that the process X_t hits b before it hits a. The Markov property shows that

$$p_+(x) = \mathrm{E}^x(p_+(X_s)) + \mathrm{O}(\mathrm{P}^x(|X_s - x| > \epsilon))$$

and from equation (6) it follows that $s^{-1}(\mathrm{P}^x(|X_s - x| > \epsilon)) \to 0$ when $s \downarrow 0$ if $a + \epsilon < x < b - \epsilon$. A Taylor expansion argument, more precisely based on the Itô formula

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)dX_s + \frac{1}{2}\int_0^t F''(X_s)\sigma^2(X_s)ds, \tag{9}$$

then shows that $p_+(x)$ must satisfy the backward equation

$$\frac{1}{2}\sigma^2(x)p_+''(x) + \mu(x)p_+'(x) = 0 \tag{10}$$

for $x \in (a,b)$. This homogeneous equation with boundary conditions $p_+(a) = 0$ and $p_+(b) = 1$ can be explicitly solved as

$$p_+(x) = \frac{u(x) - u(a)}{u(b) - u(a)}, \tag{11}$$

where, with q being a fixed, but arbitrary point in the state interval, we define

$$u(x) = \int_q^x \exp\{-\int_q^t 2\mu(z)\sigma^{-2}(z)dz\}dt. \tag{12}$$

Notice that that the same function $u(x)$ can be used for all intervals $[a,b]$. This function is called the *scale function* for the following reason: Look at the process (U_t) defined by

$$U_t = u(X_t). \tag{13}$$

By Itô's formula one easily shows that this process has no drift ($\tilde{\mu} = 0$); it is therefore said to be on *natural scale*. The diffusion coefficient of the process U_t is given by

$$\tilde{\sigma}^2(u) = \sigma^2(x)(u'(x))^2 \quad \text{with} \quad u = u(x). \tag{14}$$

In a similar way let T_{ab} be the time it takes until a or b are reached for the first time, and define

$$e(x) = \mathrm{E}^x(T_{ab}).$$

Then the Markov property shows that $a+\epsilon < x < b-\epsilon$ and $s \downarrow 0$ imply

$$e(x) = s + \mathrm{E}^x(e(X_s)) + \mathrm{O}(\mathrm{P}^x(|X_s - x| > \epsilon)).$$

Dividing by s and letting s tend to 0 gives (using Itô's formula on $e(X_s)$)

$$\frac{1}{2}\sigma^2(x)e''(x) + \mu(x)e'(x) = -1.$$

This equation can be solved by standard Green function techniques. Specifically

$$e(x) = \int_a^b G_{ab}(x,y)m(dy), \tag{15}$$

where

$$G_{ab}(x,y) = \frac{2(u(x)-u(a))(u(b)-u(y))}{u(b)-u(a)} \quad \text{for} \quad a \le x \le y \le b \tag{16}$$

and $G_{ab}(y,x) = G_{ab}(x,y)$, and where

$$m(dy) = \sigma^{-2}(y)\exp\{\int_q^y 2\mu(z)\sigma^{-2}(z)dz\}dy. \tag{17}$$

The continuity requirements needed to use Itô's formula are verified in hindsight for the solution. Specifically, assume that μ and σ are continuous and $\sigma^2(x)$ bounded away from 0 in the interior of the state interval. Then the density $\frac{m(dy)}{dy}$ of the speed measure will have continuous derivative and the scale function u will have a continuous second derivative. Weaker requirements, for instance piecewise continuity of μ and σ, are sufficient for the more general conditions given in Breiman (1968).

In the same spirit, the formulae for $p_+(x)$ and $e(x)$ will also be valid in the next Section, where no Lipschitz conditions are assumed.

3 The construction of a one-dimensional diffusion

Let μ and σ be continuous on a state interval whose interior is (c,d) (we may have $c=-\infty$ and/ or $d=+\infty$). It is also assumed that $\sigma^2(x) > 0$ on (c,d). The process we construct in this Section, will be killed if one of the boundary points is reached, so we need an extra point Δ ('coffin space') for the killed process, if necessary. Other behavior at the boundaries will be treated in the next two Sections.

It is easiest to start the construction on natural scale, so let u be given by (12), and put $(\tilde{c},\tilde{d}) = (u(c),u(d))$. Let (W_t) be a Wiener process started at $\tilde{x} \in (\tilde{c},\tilde{d})$, and let $(\mathcal{G}_t)$ be an increasing sequence of σ-algebras to which (W_t) is adapted, and which is such that $W_t - W_s$ is independent of $\mathcal{G}_s$ whenever $t > s \geq 0$. One possible choice is the σ-algebra generated by (W_t) itself, but $(\mathcal{G}_t)$ can also be much bigger, and in some cases it is very useful to have this option.

Now let Q be the (possibly infinite) random variable defined by $Q = \inf\{t : W_t = \tilde{c} \text{ or } W_t = \tilde{d}\}$, and put

$$R = \int_0^Q \tilde{\sigma}^{-2}(W_s)ds, \tag{18}$$

where $\tilde{\sigma}$ is given by (14).

For $t < R$ define a new time scale $\tau(t)$ by

$$t = \int_0^{\tau(t)} \tilde{\sigma}^{-2}(W_s)ds, \tag{19}$$

and a time-transformed Wiener process by

$$U_t = W_{\tau(t)} \quad \text{for } 0 \leq t < R \tag{20}$$

and $U_t = \Delta$ for $t \geq R$. For each t the variable $\tau(t)$ will be a stopping time for (W_t) (i.e., $[\tau(t) \leq s] \in \mathcal{G}_s$ for all $s \geq 0$), and the process (U_t) is on natural scale.

Finally we note that the scale function (12) is strictly increasing and hence has a welldefined inverse v, so we can define

$$X_t = v(U_t). \tag{21}$$

(It is understood that $X_t = \Delta$ for $t \geq R$.)

Theorem 1

The process X_t is a Markov process adapted to the σ-algebra $(\mathcal{F}_t)$ given by $\mathcal{F}_t = \mathcal{G}_{\tau(t)}$. It is continuous for $0 \leq t < R$.

The process starts at $x = v(\tilde{x})$ and has state space $(c, d) = (v(\tilde{c}), v(\tilde{d}))$.

It is a diffusion process in the sense that for all $x \in (c, d)$ the three conditions (4), (5) and (6) are satisfied.

On the other hand; assume that (4), (5) and (6) hold for some process (X_t). Then a scale function u and a speed measure $m(dy)$ can be found such that that the inverse of the construction described above gives a Wiener process, so that (X_t) has the representation (18) - (21).

The proof of this theorem is fairly straightforward using the definitions given, and will be omitted. The converse part of the Theorem is partly proved by using the reasoning of Section 2 above.

If (W_t) is the Wiener process used in the above (direct part) construction, another use of Itô's formula shows that

$$X_t(\omega) = x + \int_0^t \mu(X_s(\omega))ds + \int_0^t \sigma(X_s(\omega))dW_s. \tag{22}$$

The *existence* of a Wiener process for which (22) holds means per definition (see for instance Kallianpur (1980)) that (X_t) is a *weak* solution of the basic stochastic differential equation (1).

Theorem 2

The constructed process (X_t) will always be a weak solution of the basic stochastic differential equation. If (W_t) is the same process as in (1), then we have a strong solution.

The weak solution is unique if we limit ourselves to processes (X_t) that are killed at the boundaries of some compact subinterval of the state space.

Proof

Uniqueness of the weak solution when μ and σ^2 are bounded and continuous and when also σ^2 is bounded away from 0, was proved in great generality by Stroock and Varadhan (1969), Theorem 6.2.

$\diamond$

The uniqueness statement here can be improved by combining it with the boundary discussion which follows.

4 Boundary conditions on natural scale

To simplify the formulas, we will assume in this Section that $\mu(x) \equiv 0$, so that the process (X_t) is on natural scale with $u(x) \equiv x$. This simplifies the formula for the Green function, for instance. The interior of the state interval is still called (c,d).

The first question of interest is of course whether or not the boundary points c and/ or d can be reached by the process in finite time. A boundary point that can be reached in finite time with positive probability, is called *accessible.* An infinite boundary point can not be accessible for a process on natural scale. The following result is proved in Breiman (1968), Proposition 16.43:

Proposition 3

If d is a finite endpoint of the state interval, then d is accessible if and only if

$$\int_b^d (d-y)\sigma^{-2}(y)dy < \infty$$

when b is finite in the interior of the state interval.

Proof

The main point of the proof is that the condition is equivalent to the statement that the expected exit time T_{bd} from (b,d) will be finite from any $x \in (b,d)$. This can be seen from the formulae given in Section 2. In fact, the speed measure on natural scale is $\sigma^{-2}(y)dy$, and the Green function for the interval (b,d) is proportional to $(x-b)(d-y)$ for $y \geq x$, to $(y-b)(d-x)$ for $y \leq x$. In fact, if the integral of Proposition 3 is infinite, then by equation (15) we have for $x \in (b,d)$

$$\mathrm{E}^x(T_{bd}) \geq \text{const.} \cdot (x-b) \int_x^d (d-y)\sigma^{-2}(y)dy = +\infty.$$

(See also Lemma 4 below.) Conversely, assume that the integral is finite for one $b \in (c,d)$ and assume at teh same time that d is inaccessible. Then the expected time to reach b from $x \in (b,d)$ will be

$$\mathrm{E}^x(T_b) = \lim_{z\uparrow d}\mathrm{E}^x(T_{bz}) = \int_b^d G_{bd}(x,y)\sigma^{-2}(y)dy.$$

The lefthand side of this is obviously nondecreasing as $x \uparrow d$. On the other hand the integral is

$$\frac{2(d-x)}{d-b}\int_b^x (y-b)\sigma^2(y)dy + \frac{2(x-b)}{d-b}\int_x^d (d-y)\sigma^{-2}(y)dy.$$

This tends to zero as $x \uparrow d$, which gives a contradiction.

◇

The first part of the proof needs to be completed at a point which turns out to be of further interest. By contraposition we proved above that the integral was finite when $\mathrm{E}^x(T_{bd})$ was finite, while we really wanted this to follow from an assumption that T_{bd} was finite with positive probability. Since the distribution of T_{bd} must depend upon the starting point x, one way to make this precise is to assume that $\sup_{x\in(b,d)}\mathrm{P}(T_{bd} > t) < 1$ for some $t > 0$. More generally it turns out that the *expected* exit time from intervals plays such a central rôle in this proof and in the proof of several results below while the definition of accessibility etc. just involves the *probability* of exit. A way around this problem lies in the following Lemma, which turns out to be useful for all these proofs.

Lemma 4

Let T_J be the first exit time from an interval in the state space for (X_t). Assume $\sup_{x\in J}\mathrm{P}^x(T_J > t) = \alpha < 1$ for some $t > 0$. Then

$$\sup_{x\in J}\mathrm{E}^x(T_J) \leq \frac{t}{1-\alpha}.$$

<u>Sketch of proof:</u>

Use an induction argument to show that $\sup_{x\in J}\mathrm{P}^x(T_J > nt) \leq \alpha^n$. Then use the familiar formula $\mathrm{E}^x(T_J) = \int_0^\infty \mathrm{P}^x(T_J > \tau)d\tau$ to bound the expectation above by a geometric series. ◇

The accessible boundary points are of two different types. For the *exit* boundaries the only possible further behaviour after the boundary is reached, is absorption. On the other hand we have the *regular* boundary points, exemplified by finite boundaries that may be imposed on a Wiener process. For the regular boundaries several possibilities exist for further behaviour, the most interesting being absorption and reflection. Thus, to classify an accessible boundary point further, one may look at the possibility of reflection in the point. This can be studied by adding the mirror image of the state interval around the boundary point and trying to start the process again in this point. Again it is crucial if the expected exit time from a small interval is finite, and the answer is:

Proposition 5
The accessible boundary point d is regular if and only if $\int_b^d \sigma^{-2}(y)dy < \infty$*; otherwise it is exit.*

The inaccessible boundaries are also of two types. The boundary d is said to be of *entrance* type if it is possible to start the process in d (a limiting argument must be used if d is infinite) and then reach the interior of the state interval, otherwise it is called *natural.* Again a similar reasoning (Breiman, 1968, Proposition 16.45) gives:

Proposition 6
An inaccessible boundary point d is natural if and only if

$$\int_b^d y\sigma^{-2}(y)dy = +\infty$$

for b in the interior of the state interval, otherwise it is entrance.

The natural boundaries is a rather wide class. One further property that may be tied to these together with the regular (reflecting) boundary points is that a stationary probability density exists for the process if and only if the speed measure $\int_c^d \sigma^{-2}(y)dy$ of the whole state interval is finite. Another interesting class of natural boundaries are the *attracting* ones. The upper point d is attracting if there is a positive probability that X_t shall converge to d as $t \to \infty$.

Proposition 7
A natural boundary point d is attracting if and only if $d < +\infty$ *and* $\int_b^d \sigma^{-2}(y)dy = +\infty$ *for* $b \in (c, d)$.

5 General boundary conditions

The translation of the results of the previous Section is straightforward. The definitions of the boundary types (inaccessible, accessible, regular, exit, entrance, natural and attracting) are the same, and the scale function and the speed measure are defined by (12) and (17), respectively, repeated here for convenience

$$m(dy) = \sigma^{-2}(y)\text{exp}(\int_b^y 2\mu(z)\sigma^{-2}(z)dz)dy,$$

$$u(x) = \int_b^x \text{exp}(-\int_b^t 2\mu(z)\sigma^{-2}(z)dz)dt.$$

Here b is an arbitrary point in the interior (c, d) of the state interval. The classification results can be summarized as follows:

Theorem 8

A. A necessary and sufficient condition for d to be accessible is that $u(d) < \infty$ *and* $\int_b^d (u(d) - u(y))m(dy) < \infty$.

B. An accessible boundary point d is regular if and only if $\int_b^d m(dy) < \infty$. *Otherwise it is an exit boundary.*

C. An inaccessible boundary point d is natural if and only if $\int_b^d u(y)m(dy) = \infty$.

D. A natural boundary point d is attracting if and only if $u(d) < \infty$ *and at the same time* $\int_b^d m(dy) = \infty$.

Some comments to this basic Theorem may be in order. First of all, we have only proved part A in detail. The details of the other proofs are different, but they are not very different in spirit. Secondly, to verify the different conditions, one has to investigate the finiteness of an integral which can be fairly complicated in concrete cases. Simplification of this process is sometimes possible. For instance, one can see that d is a regular boundary point when both $u(d)$ and $\int_b^d m(dy)$ are finite. Also, one can show that an inaccessible boundary d with $u(d) < \infty$ must be natural and attracting.

It should be obvious how the corresponding conditions are for the lower boundary point c.

6 Some simple examples

All the examples below can be classified on the basis of Theorem 8.

a. The simplest example is the Wiener process W_t started at $x = 0$, say. Here $\mu(x) = 0$ and $\sigma(x) = 1$, and an upper boundary d is natural if $d = +\infty$, regular if $d < +\infty$. With a constant drift $\mu \neq 0$ the situation is similar, with the addition that $d = +\infty$ is attracting when $\mu > 0$ and $c = -\infty$ is attracting when $\mu < 0$.

b. It should be clear how these results transfer to functions of the Wiener process, e.g., $X_t = \exp(W_t)$, which has $\mu(x) = \frac{1}{2}x$ and $\sigma(x) = x$. Here the lower boundary 0 will be natural.

c. Let $\mu(x) = 0$ and $\sigma^2(x) = x^\alpha$, and assume that the interior of the state interval is $(0, +\infty)$. Then the lower boundary point $c = 0$ is regular for $\alpha < 1$, an exit boundary for $1 \leq \alpha < 2$ and natural (attracting) for $\alpha \geq 2$. The upper boundary $d = +\infty$ is always inaccessible. It is natural for $\alpha \leq 2$, entrance for $\alpha > 2$.

d. Assume $\mu(x) = 0$, $\sigma^2(x) = x^\alpha(1-x)^\alpha$ with α real, and take $c = 0$ and $d = 1$. Then both c and d are classified in the same way as the boundary $c = 0$ in the previous example.

e. Both boundary points $\pm\infty$ of the Ornstein-Uhlenbeck process ($\mu(x) = \gamma - \alpha x$, $\sigma^2(x) = \sigma^2$) are natural. The upper boundary $d = +\infty$ is attracting if $\alpha < 0$ or if $\alpha = 0$ and $\gamma > 0$.

f. The process specified by $\mu(x) = \alpha x$, $\sigma^2(x) = \beta x$ ($\beta > 0$), $c = 0$ and $d = +\infty$ may be used as an approximation for branching processes. The lower boundary point 0 is always exit (and hence absorbing), and the upper boundary $+\infty$ is always natural. It is attracting if $\alpha > 0$.

g. Finally, several of the diffusion models used in genetics may be subsumed under the parametric class specified by $\mu(x) = \theta - (\phi + \theta)x + \psi x(1-x)(\eta + x(1-2\eta))$ with $\sigma^2(x) = \frac{1}{2}x(1-x)$, $c = 0$ and $d = 1$. The process (X_t) will then describe (approximately) a gene frequency, and the time scale is such that a generation is a very small time interval. The nonnegative parameters ϕ and θ correspond to mutation frequencies, and ψ and η are related to selection. The lower boundary $c = 0$ is exit when $\theta = 0$, regular (reflecting) when $0 < \theta < \frac{1}{4}$ and entrance (hence inaccessible) when $\theta \geq \frac{1}{4}$. The upper boundary $d = 1$ is classified similarly based upon the parameter ϕ.

7 Population growth in a stochastic environment

Lungu and Øksendal (1996) have recently proposed the following stochastic differential equation to describe the growth of a populaton in a stochastic environment with finite carrying capacity $k > 0$:

$$dX_t = rX_t(k - X_t)dt + \alpha X_t(k - X_t)dW_t.$$

This is a diffusion process with $\mu(x) = rx(k-x)$, $\sigma(x) = \alpha x(k-x)$ and natural state interval from $c = 0$ to $d = k$. Ley b be a point in the interior of this state interval. Then the scale function is

$$u(x) = \int_b^x \left(\frac{(k-y)b}{y(k-b)}\right)^\beta dy = \text{const.} \cdot \int_b^x \left(\frac{k}{y} - 1\right)^\beta dy, \quad \text{where } \beta = \frac{2r}{\alpha^2 k}. \qquad (23)$$

Similarly, the speed measure is

$$m(dy) = \left(\frac{1}{\alpha y(k-y)}\right)^2 \left(\frac{y(k-b)}{(k-y)b}\right)^\beta dy = \text{const.} \cdot y^{\beta-2}(k-y)^{-\beta-2}dy. \qquad (24)$$

On this basis the classification of boundary points according to Theorem 8 runs: *Both boundaries* 0 *and* k *are inaccessible, and furthermore, they are always natural. If* $\beta > -1$ *(i.e,* $r > -\frac{\alpha^2 k}{2}$*), the boundary* k *is attractive. If* $\beta < 1$ *(i.e.,* $r < \frac{\alpha^2 k}{2}$*), the boundary* 0 *is attractive.*

Lungu and Øksendal (1996) proved using other means that when only k is attractive, the process X_t converges almost surely to k when $t \to \infty$, and that when only 0 is attractive, then the process converges almost surely to 0 when $t \to \infty$. These results hold whatever the value $X_0 = x$ has in the interval $(0, k)$, and they are not unexpected. What remains is the case where *both* boundaries are attractive, i.e., when $-1 < \beta < 1$, which means $-\frac{\alpha^2 k}{2} < r < \frac{\alpha^2 k}{2}$. We will show that in this case the process converges to k with probability $p_+(x)$, where $x = X_0$ and $p_+(x)$ is given by (11) with $a = 0$ and $b = k$, in this case

$$p_+(x) = \frac{\int_0^x y^{-\beta}(k-y)^\beta dy}{\int_0^k y^{-\beta}(k-y)^\beta dy}.$$

Furthermore, the process converges to 0 with probability $1 - p_+(x)$. These results hold quite generally.

Theorem 9

Assume that a diffusion process (X_t) *is started at* $x \in (c, d)$*, the interior of the state interval, and assume that both boundaries* c *and* d *are attractive (hence natural and inaccessible). Let* $p_+(x)$ *be given by (11) with* $a = c$ *and* $b = d$*. Then* X_t *converges with probability* $p_+(x)$ *to* d *and with probability* $1 - p_+(x)$ *to* c*.*

Proof

We will prove the first statement; the second is proved similarly. It is essential that $u(c)$ and $u(d)$ are finite, so that p_+ is continuous in c and d. Thus $\mathrm{P}^x[X_t$ reaches $d-\delta$ before $c+\delta]$ is arbitrarily close to $p_+(x)$ when δ is small enough. By the strong Markov property the proof will be complete if we can show that for any $\epsilon > 0$ we can take δ so small that $\mathrm{P}^{d-\delta}[\inf_t X_t \le d - \epsilon] \le \epsilon$.

Let now $\epsilon > 0$ be given. The proof will go by defining two sequences $\{\delta_n\}$ and $\{\eta_n\}$, both tending monotonically to 0 and such that $\epsilon > \eta_1 > \delta_1$. First we use the fact that the scale function $u(x)$ is bounded as $x \uparrow d$. From this fact follows that

$\sup_{0<\delta<\eta_n} \mathrm{P}^{d-\eta_n}[d-\epsilon$ is reached before $d-\delta] \to 0$ as $\eta_n \to 0$. We can therefore choose η_n so small that $\mathrm{P}^{d-\eta_n}[d-\epsilon$ is reached before $d-\delta_{n+1}] \le 2^{-n-1}\epsilon$, and this choice can be made *independent of the choice of* δ_{n+1}. Then it also follows that $\mathrm{P}^y[d-\epsilon$ is reached before $d-\delta_{n+1}] \le 2^{-n-1}\epsilon$ whenever $y \ge d-\eta_n$.

Secondly, given n and η_n it always possible to choose $\delta_n(<\eta_n)$ so small that $\mathrm{P}^{d-\delta_n}[T_{d-\eta_n} \le 1] \le 2^{-n-1}\epsilon$, where $T_{d-\eta_n}$ is the first time $d-\eta_n$ is hit. This is possible since d is a natural boundary, and it then follows (Breiman, 1968, p. 366) that $\lim_{y\to d}\mathrm{P}^y[T_x < t] = 0$ for all t.

Taking $\delta = \delta_1$ and making repeated use of the strong Markov property, this construction shows that

$$\mathrm{P}^{d-\delta}[\inf_t \le d-\epsilon] \le \sum_{n=1}^{\infty} 2^{-n-1}\epsilon + \sum_{n=1}^{\infty} 2^{-n-1}\epsilon = \epsilon ,$$

which completes the proof. $\diamond$

8 Limit theorems

Virtually every mathematical model must be regarded as an approximation, and this is also true for stochastic models. It is of considerable importance to find criteria under which more complicated models can be approximated by simpler ones. For diffusion processes there is a rough way to state these criteria: The initial distributions must be about right, and in addition the infinitesimal conditions (4), (5) and (6) must be 'approximately' satisfied.

A precise statement is as follows: Let (X_t) be a diffusion as presented here *without regular boundaries*. Let $\{X^n\} = \{(X_t^n)\}$ be a sequence of processes (right-continuous with left-hand limits) such that each (X_t^n) is adapted to an increasing family of sigmafields $(\mathcal{F}_t^n)$. Technically the mode of convergence considered will be weak convergence in the space $D[0,\infty)$ of right-continuous functions with left-hand limits endowed with the Stone topology, which is the usual concept of convergence in distribution of processes. In practice this means convergence of all finite-dimensional distributions, and in addition convergence in distribution of functionals like $f(X^n) = \sup_{0\le t\le a}|X_t^n|$ etc.

Let $0 = t_0^n \le t_1^n \le \cdots; \lim_{k\to\infty} t_k^n = +\infty$ almost surely. Furthermore let $r^n(t) = \max\{k \ge 0 : t_k^n \le t\}$, and define $\Delta t^n(k) = t_{k+1}^n - t_k^n$. Assume that for each $t > 0$ we have $\max_{0\le k\le r^n(t)} \Delta t^n(k) \to 0$ in probability as $n \to \infty$.

The Theorem below give conditions for convergence to a diffusion process that are weakest possible in some sense, and these conditions are reminiscences of (4), (5) and (6). We use the abbreviation $\Delta_1 X^n(k) = (X^n_{t^n_{k+1}} - X^n_{t^n_k}) \cdot I(|X^n_{t^n_{k+1}} - X^n_{t^n_k}| \le 1)$.

Theorem 10
Assume that X_0^n converges to X_0 in distribution and that for all ϵ, $t > 0$ we have

$$\sum_{k=0}^{r^n(t)} \mathrm{P}[\sup_{t_k^n \le s \le t_{k+1}^n} |X_s^n - X_{t_k^n}^n| > \epsilon | \mathcal{F}_{t_k^n}^n] \to 0 \text{ in probability,}$$

$$\mathrm{P}[\inf_{0 \le s \le t} X_s^n < c - \epsilon] \to 0$$
$$\mathrm{P}[\sup_{0 \le s \le t} X_s^n > d + \epsilon] \to 0.$$

Furthermore, assume that for all $t > 0$ and for all compact intervals K in the interior (c, d) of the state intervall we have

$$\sum_{k=0}^{r^n(t)} |\mathrm{E}(\Delta_1 X^n(k) | \mathcal{F}_{t_k^n}^n) - \mu(X_{t_k^n}^n) \Delta t^n(k)| I(X_{t_k^n}^n \in K) \to 0 \text{ in probability}$$

$$\sum_{k=0}^{r^n(t)} |\mathrm{E}((\Delta_1 X^n(k))^2 | \mathcal{F}_{t_k^n}^n) - \sigma^2(X_{t_k^n}^n) \Delta t^n(k)| I(X_{t_k^n}^n \in K) \to 0 \text{ in probability.}$$

Then the sequence of processes $\{X^n\}$ converges weakly in $D[0,\infty)$ to the process X.

Of course these conditions for convergence can be simplified in most cases of interest. If (X_t^n) is a Markov process, then the conditional expectations and probabilities will be much simplified. If a Lindeberg type condition holds, then moments of ordinary increments can be used instrad of truncated increments. If we know that each of the processes X^n is confined to the interval $[c, d]$, then the boundary conditions may be dropped.

For some applications it will represent a clear simplification in itself that the proof of convergence in the two moment conditions can be limited to compacts in the interior of the state interval.

The proof of Theorem 10 is given in Helland (1981), and while there are details to work through, the idea of the proof is very simple: Sets of conditions for convergence towards the Wiener process are well known; these represent in fact thoroughly studied variants of the classical central limit theorem. Specifically, central limit theorems of the martingale type contain condition for convergence that are very close to those of Theorem 10. Now it is shown in the present paper that all time homogeneous one-dimensional diffusion processes can be represented as a time-transformed, scale-changed Wiener process. If we can show that the transformations involved are continuous and if we are able to transform the conditions for convergence, we have a receipt for deriving theorems of the type exeplified in Theorem 10. This was in fact the program that was followed in Helland (1981): In particular, the continuity of the time transformation was studied in detail in Helland (1978).

In what way, then, are the condition for convergence given here minimal? They are not minimal for weak convergence in $D[0,\infty)$, it turns out, but for a stronger form of convergence, namely joint convergence in distribution of all sets of conditional distributions, given the past. This convergence is implied by the conditions, and on the other hand, if this form of convergence holds, it is alway possible to find a sequence of partions such that the conditions of Theorem 10 hold. This is studied in details for convergence towards the Wiener process in Helland (1980), in general in Helland (1981).

Finally, we have excluded regular boundary point from the above discussion. These cases require a special treatment. (This seems to be one of the very few cases in mathematics where *regular* points cause extra difficulties.) A detailed discussion is given in Helland (1982), where differents sets of conditions for convergence are given, both for the case of an absorbing boundary and for the case of a reflecting boundary.

Thus a fairly complete theory for the one-dimensional homogeneous case can be given, and it is based upon the same simple idea that was used here to characterize the processes themselves without needing regularity conditions and for a complete classification of boundaries: Transforming the Wiener process in space and time.

Of course, in many cases we need processes that are time inhomogeneous or processes in several dimension. Then stronger mathematical tools are needed. For the processes themselves we have already referred to Øksendal (1995); see also references there. For general limit theorems resembling those given above, but requiring certain regularity conditions, see Ethier and Kurtz (1986) and references there.

References.

Breiman, L. (1968). *Probability.* Siam, Philadelphia.

Dynkin, E.B. (1965). *Markov Processes.* Springer-Verlag, Berlin.

Ethier, S.N. and Kurtz, T.G.(1986). *Markov Processes: Characterizations and Convergence.* Wiley, New York

Feller, W. (1952). The parabolic differential equations and the associated semigroups of transformations. *Annals of Mathematics* **55**, 468-519.

Feller, W. (1954). Diffusion processes in one dimension. *Trans. Amer. Math. Soc.* **77**, 1-31

Feller, W. (1957). Generalized second order differential operators and their lateral conditions. *Illinois J. Math.* **1**, 495-504.

Helland, I.S. (1978). Continuity of a class of random time transformations. *Stochastic Processes and their Applications* **7**, 79-99.

Helland, I.S. (1980). On weak convergence to Brownian motion. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **52**, 251-265.

Helland, I.S. (1981). Minimal conditions for weak convergence to a diffusion process on the line. *The Annals of Probability* **9**, 429-452.

Helland, I.S. (1982). Convergence to diffusions with regular boundaries. *Stochastic Processes and their Applications* **12**, 27-58.

Itô, K. and McKean, H.P. (1965). *Diffusion Processes and Their Sample Paths.* Springer-Verlag, Berlin.

Kallianpur, G. (1980). *Stochastic Filtering Theory.* Springer-Verlag, Berlin

Lungu, E.M. and Øksendal, B. (1996). Optimal harvesting from a population in stochastic crowded environment. *Preprint*

Stroock and Varadhan (1969). Diffusion processes with continuous coefficients, I. *Communications on Pure and Applied Mathematics* **22**, 345-400.

Øksendal, B. (1995). *Stochastic Differential Equations.* Springer-Verlag, Berlin.